WAVE MOTIONS OF A FLUID DUE TO A MOVING SYSTEM OF PRESSURES ON AN INCLINED BOTTOM

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In this paper we obtain solutions of problems concerning waves generated by a system of moving pressures applied to a free surface. We carry out an asymptotic analysis and study structures of the wave fields.

The problem concerning nonstationary wave motions of a fluid under the action of a moving system of periodic surface pressures was solved in [1, 2] for a region of constant depth. We consider this problem for a region of variable depth, using a method presented in [3] in connection with a Cauchy–Poisson problem in which a mixed spectrum of characteristic values (continuous and discrete portions) was established and an expansion theorem was proved.

The equations, boundary and initial conditions for the problem presented in [1, 2] are as follows:

$$\Delta\varphi = 0,\ 0<r<\infty,\ -\beta<\theta<0,\ |z|<\infty,\quad \beta=\pi/2n,\ n=2m+1,\quad m=0,1,2,\ldots; \tag{1}$$

$$\varphi_{tt}+\frac{g}{r}\varphi_\theta=-\frac{1}{\rho}P_t,\ \theta=0;\ \varphi_\theta=0,\ \theta=-\beta;\ \varphi=0,\ \varphi_t=-\frac{1}{\rho}P_0,$$

$$\theta=0,\quad t=0;\ \varphi<\infty,\ r\to0;\ \varphi\to0,\ \sqrt{r^2+z^2}\to\infty;$$

$$\eta=-\frac{1}{g}\varphi_t|_{\theta=0}-\frac{P}{\rho g},\quad \Delta\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.$$

Here ϕ is the velocity potential; r, θ, z are cylindrical coordinates; t is the time; g is the free-fall acceleration; ρ is the fluid density; P and P_0 are the pressures at an arbitrary instant of time and at the initial instant, respectively; η is the form of the free surface (Fig. 1).

We examine the wave motion generated by a periodic moving system of surface pressures propagating in the direction of the normal to the curve of the shore and applied in a semi-infinite strip of width 2a, i.e.,

$$P(x,z,t)=P^0 f(z)\exp\{i(kx-\omega t)\},\quad f(z)=f^0\begin{cases}1, & |z|\leqslant a,\\ 0, & |z|>a.\end{cases}$$

We construct a solution of problem (1) using the method of integral transforms [3]; however, since the expansion theorem is valid for absolutely integrable functions, we introduce a regularizing factor $(-\alpha x)$ ($\alpha > 0$) for function P(x, z, t). After effecting the transformations and passing to the limit as $\alpha \to 0$, we obtain the expression $\eta = -P/\rho g + \eta^c + \eta^d$, for waves on the free surface, where η^c and η^d are stipulated by the characteristic functions of the continuous and discrete spectra, respectively:

$$\eta^c=\frac{P^0}{\pi\rho g}\int_0^{\pi/2}D\left\{\int_0^\infty\frac{1}{\cos\lambda}\sin(s_c a\cos\lambda)\cos(s_c z\cos\lambda)\times\right.$$
$$\times\exp(A_0 r)D\left(\frac{I}{A_0+ik}\right)\left[\frac{2\omega^2}{\omega^2-gs_c}\exp(-i\omega t)+\frac{\sqrt{gs_c}}{\omega+\sqrt{gs_c}}\exp i\sqrt{gs_c}\,t-\right.$$
$$\left.\left.-\frac{\sqrt{gs_c}}{\omega-\sqrt{gs_c}}\exp\left(-i\sqrt{gs_c}\,t\right)\right]ds_c\right\}d\lambda; \tag{2}$$

$$\eta^d=\frac{P^0}{\pi\rho g}\sum_{l=1}^{n-1}\varepsilon_l D_l\left\{\int_0^\infty\sin pa\cos pz\exp(A_l r)\times\right. \tag{3}$$

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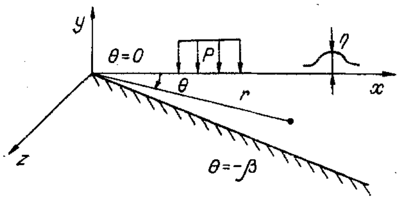

Fig. 1

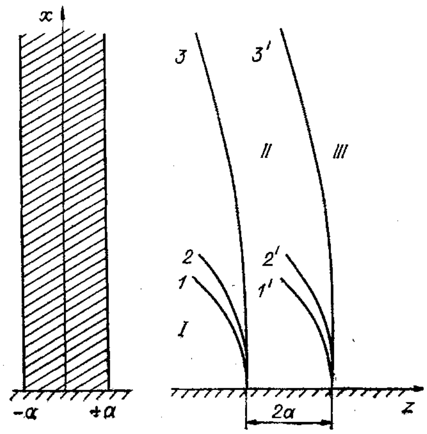

Fig. 2

$$\times D\left(\frac{1}{A_l+ik}\right)\left[\frac{2\omega^2}{\omega^2-gp\cos l\beta}\exp(-i\omega t)+\frac{\sqrt{gp\cos l\beta}}{\omega+\sqrt{gp\cos l\beta}}\times\right.$$
$$\left.\times\exp i\sqrt{gp\cos l\beta}\,t-\frac{\sqrt{gp\cos l\beta}}{\omega-\sqrt{gp\cos l\beta}}\exp\left(-i\sqrt{gp\cos l\beta}\,t\right)\right]dp\Bigg\}.$$

Here D and D_ℓ are the operators

$$D=\frac{1}{2\sqrt{2\pi}}\sum_{j,\chi=0}^{1}\sum_{k=1}^{n}B_k^{(\chi)},\quad D_l=\frac{1}{2\sqrt{2}}\sum_{j=0}^{1}\sum_{k=1}^{n-1}B_{kl}.$$

We give explicit formulas for the coefficients in the solution of problem (2), (3) in the Appendix.

It was shown in [3] that the wave motion of a fluid over an inclined bottom is characterized by the presence of waves of continuous and discrete spectra, whereby the waves of continuous spectrum predominate at the equator far from the shoreline while those of the discrete spectrum are localized in the coastal zone.

In studying waves of the continuous spectrum we employ the stationary phase method (the large parameter is defined by the formula $W = \omega^2 R_\alpha/g$ ($R_\alpha = \sqrt{r^2 + (z + (-1)^\alpha a)^2}$, $\alpha = 0;\ 1$)) and the Cauchy residue theorem for those integrals containing poles [2, 4]. As a result, at some distance from the region of pressures, we have the following for the form of the free surface:

$$\eta^c \simeq P^0 f^0\frac{\omega}{\rho g}\sqrt{\frac{2\pi}{g}}\,i\exp(-i\omega t)\sum_{\alpha=0}^{1}\frac{(-1)^\alpha}{z+(-1)^\alpha a}\times \tag{4}$$
$$\times D^0\left[R_{kj}^{1/2}D\left(\frac{I}{A_0^*+ik}\right)\exp\left(-\frac{\omega^2}{g}r\sin a_{kj}+i\left(\frac{\omega^2}{g}R_{kj}-\frac{\pi}{4}\right)\right)\right]$$

for $R_{kj} < u_0 t$ (operator D^0 is defined in Sec. 2 of the Appendix),

$$\eta^c = O\left(R_\alpha^{-2}\right)\ \text{for}\quad R_{kj} > u_0 t,$$

where

$$R_{kj}=\sqrt{r^2\cos^2 a_{kj}+\left(z+(-1)^\alpha a\right)^2};\quad u_0=\frac{g}{2\omega};$$

$$A_0^*=-\frac{\omega^2}{g}\left(\sin a_{k'j'}+(-1)^{\chi'}\frac{r}{R_{kj}}\cos a_{kj}\cos a_{k'j'}\right).$$

Thus, motion of the fluid, stipulated by waves of the continuous spectrum, is divided into two groups, consisting of m + 1 families, each situated at a distance of 2a from one another. A family in a group is characterized by a definite value of the expression $|\cos a_{kj}|$. The wave field described by a family is separated into two regions. The boundary between them represents a portion of an ellipse with equation $R_{kj} = u_0 t$ and with minor axis on the shoreline at the points $z = \pm a + u_0 t$. This elliptical curve moves with speed u_0, staying similar to itself and receding from the region of application of the system of pressures. All the elliptic curves exit from two moving points on the shore, $z = \pm a + u_0 t$, and are tilted to the side of the $z = 0$ axis. For the family characterized by the expression $|\cos a_{kj}| = 1$, the elliptic curve converts into a portion of a circle. A family with $|\cos a_{kj}| = 1$ is present for an arbitrary angle of inclination β of the bottom; when $\beta = \pi/2$, which corresponds to an infinitely deep fluid, this family completely defines the wave picture of the motion. As β decreases, the number of elliptic curves increases and their eccentricity tends to increase. As $\beta \to 0$, the number of curves increases without bound and the outer curve tends towards a halfline parallel to the x axis. Inside a region bounded by an elliptic curve there propagate progressive waves, which decay according to the law $R_\alpha^{-1/2}$ for small r and according to an exponential law for large r. Outside of this region there propagate rapidly decaying waves according to the law R_α^{-2} for small r and according to an exponential law for large r. For a family with $|\cos a_{kj}| = 1$ exponential factors in the decay exponent are equal to 1, and inside a region bounded by an arc of the circle $\sqrt{r^2 + (z + (-1)^\alpha a)^2} = u_0 t$ there propagate progressive waves decaying according to the law $R_\alpha^{-1/2}$, and outside this region according to the law R_α^{-2}.

The wave picture resulting from superimposing $2(m + 1)$ families can be characterized as follows: in region I, bounded by an arc of the circle $r^2 + (z - a)^2 = u_0^2 t^2$ and the shoreline, at a sufficient distance from the region of pressures and the shoreline, there propagate progressive waves decaying according to the law $R_\alpha^{-1/2}$; in region III, being the exterior of the elliptic curve $r^2 \sin^2\beta + (z + a)^2 = u_0^2 t^2$, there propagate progressive waves, decaying rapidly (according to the law R_α^{-2}); region II represents a transition regime from waves of one type to waves of the other type.

Figure 2 shows boundaries of the regions for $\beta = \pi/10$. In this case $m = 2$ and the wave picture consists of the superposition of two groups, each containing up to three families of waves. To each family there corresponds a moving boundary: either a portion of an elliptic curve (curves 2, 2', 3, 3') or an arc of a circle (curves 1, 1'). Region I is bounded above by an arc of a circle, while region III is bounded on the left by a portion of an elliptic curve.

We proceed now to study the wave motion stipulated by waves of the discrete spectrum. We consider the coastal zone characterized by small r. Studying η^d, we find, analogous to waves of the continuous spectrum,

$$\eta^d = \frac{P^0 f^0}{4\pi\rho g} \sum_{l=1}^{n-1} \varepsilon_l \eta_l^d. \tag{5}$$

Here η_ℓ^d may be calculated from the formulas

$$\eta_l^d = \sum_{\alpha=0}^{1} (-1)^\alpha I_\alpha \quad \text{for} \quad z < -a + u_l t,$$

$$\eta_l^d = -I_1 \quad \text{for} \quad -a + u_l t < z < a + u_l t, \quad \eta_l^d = O\left(z^{-5/2}\right) \quad \text{for} \quad z > a + u_l t,$$

$$I_\alpha \simeq 4\pi i \frac{\omega^2}{g \cos l\beta} D_l\left(\frac{1}{A_l^* + ik}\right) D_l(1) \exp\left\{i\left(\frac{\omega^2}{g \cos l\beta}(z + (-1)^\alpha a) - \omega t\right)\right\},$$

$$A^* = -\frac{\omega^2}{g \cos l\beta} \sin(a_{kj} + l\beta), \quad u_l = \frac{g \cos l\beta}{2\omega}.$$

As a result of superposition of all the components, we obtain the following wave picture: η^d is calculated from formula (5), where $\eta_l^d = \sum_{\alpha=0}^{1} (-1)^\alpha I_\alpha$ for $z < -a + u_{n-1} t$; for $-a + u_{n-1} t < z < a + u_2 t$ we have the transition regime, namely: the gradual vanishing of components from the total sum, and $\eta^d = O(z^{-5/2})$ for $z > a + u_2 t$ ($u_2 = g \cos 2\beta/2\omega$, $u_{n-1} = g \sin \beta/2\omega$).

In the coastal zone ($r \to 0$), among the m component waves of the discrete spectrum, the defining wave is the Stokes wave ($\ell = n - 1$). The wave motion corresponding to it has the form

$$\eta^d = \frac{P^0 f^0}{4\pi\rho g} \varepsilon_{n-1} \eta_{n-1}^d. \tag{6}$$

Here η_{n-1}^d is the term stipulated by the $n - 1$ value of the discrete spectrum, given by the expressions

$$\eta^d_{n-1} \simeq 4\pi \frac{\omega^2}{\sin\beta} B^2_{1,n-1} \frac{\omega^2 \operatorname{ctg}\beta + ikg}{\omega^4 \operatorname{ctg}^2\beta + k^2g^2} \sin \frac{\omega^2}{g\sin\beta} a \exp\left\{i\left(\frac{\omega^2}{g\sin\beta} z - \omega t\right)\right\}$$
for $z < -a + u_{n-1}t$,
$$\eta^d_{n-1} \simeq 2\pi i \frac{\omega^2}{\sin\beta} B^2_{1,n-1} \frac{\omega^2 \operatorname{ctg}\beta + ikg}{\omega^4 \operatorname{ctg}^2\beta + k^2g^2} \exp\left\{i\left(\frac{\omega^2}{g\sin\beta}(z-a) - \omega t\right)\right\} \text{ for } -a +$$
$$+ u_{n-1}t < z < a + u_{n-1}t, \quad \eta^d_{n-1} = O(z^{-5/2}) \text{ for } z > a + u_{n-1}t.$$

Analogous to waves of the continuous spectrum, the fluid motion stipulated by waves of the discrete spectrum amounts to a superposition of m wave structures in the coastal zone. Each wave structure consists of waves progressing along the z axis and propagating, according to different laws, outside and inside the strip of width 2a. Each strip moves with speed $g \cos \ell\beta/2\omega$ in the direction of increasing z. We note that associated with the wave structure there is a rate of motion of the strip and all these rates are less than the rates of motion of the corresponding zones for waves of the continuous spectrum. In the region to the right of the moving line $z = a + u_2t$, there are waves of all components; in the zone to the left of the line $z = -a + u_{n-1}t$ wave motion stipulated by the applied pressure is absent.

In the coastal zone a progressive non-decaying wave propagates up to the moving point $z = a + u_{n-1}t$, after which waves, stipulated by the applied pressure, are lacking. In case $t \to \infty$ a stationary regime is formed on the free surface of the fluid, a regime characterized by the following peculiarities: along rays leaving the points $z = \pm a$, there propagate progressive diverging waves decaying according to the law $R_\alpha^{-1/2}$ and described by formula (4) in the coastal zone there propagate progressive non-decaying boundary waves described by formula (5); in the shore zone there propagates a progressive non-decaying Stokes wave described by formula (6).

We direct the curve of application of pressure parallel to the shoreline and assume that the pressure is applied in the strip, i.e., $P(x, z, t) = f(x) \exp\{i(kz - \omega t)\}$, where

$$f(x) = f^0 \begin{cases} 1, & a \leqslant x \leqslant b, \\ 0, & 0 < x < a, \quad x > b. \end{cases}$$

We seek a solution of problem (1) in the form $\phi(r, \theta, z, t) = \psi(r, \theta, k, t) \exp ikz$.

Using the method presented in [3], when p = k we obtain, for the form of the free surface $\eta = -P/\rho g + \eta^c + \eta^d$

$$\eta^c = \frac{f^0}{2\rho g} \exp i(kz - \omega t) \int_0^\infty D\left\{(\exp A_0 r) D\left(\frac{I}{A_0}(\exp A_0 b - \exp A_0 a)\right)\left[\frac{\sqrt{gs_c}}{\sqrt{gs_c} - \omega} \exp i(\omega - \sqrt{gs_c})t + \frac{\sqrt{gs_c}}{\sqrt{gs_c}+\omega} \times \exp i(\sqrt{gs_c} + \omega)t - \frac{2\omega^2}{gs_c - \omega^2}\right] dq\right\}; \tag{7}$$

$$\eta^d = \frac{f^0}{2\rho g} \sum_{l=1}^{n-1} \varepsilon_l D_l (\exp A_l r) D_l \left(\frac{\exp A_l a - \exp A_l b}{\sin(a_{kj} + l\beta)}\right) \times \left[\frac{\sqrt{gk\cos l\beta}}{\sqrt{gk\cos l\beta} - \omega} \exp i(kz - \sqrt{gk\cos l\beta})t + \frac{\sqrt{gk\cos l\beta}}{\sqrt{gk\cos l\beta}+\omega} \times \exp i(kz + \sqrt{gk\cos l\beta}\, t) - \frac{2\omega^2}{gk\cos l\beta - \omega^2} \exp i(kz - \omega t)\right]. \tag{8}$$

We make a study of waves of the continuous spectrum (7) by analogy with the problems studied in [2, 5] for a fluid of infinite depth. We write

$$\eta^c = \eta^c(b) - \eta^c(a). \tag{9}$$

Here, in a region separated a sufficient distance from the strip of application of pressure and the shoreline ($r \gg b$), $\eta^c(\kappa)$ is given by the following expressions:

1) if $k < k_1$, $k_1 = \frac{\omega^2}{g\sqrt{3}}$ $\left(k_1 < \frac{\omega^2}{g}\right)$, then

$$\eta^c(\varkappa) = \frac{f^0}{2\rho g} I_1^c \quad \text{for} \quad |M| < u^* t, \tag{10}$$

$$I_1^c \simeq 4\pi i \frac{\omega^4}{\sqrt{\omega^4 - k^2 g^2}} \exp i(kz - \omega t) D^0 \Big(D \big(I \big(\omega^2 \sin a_{k'j'} + (-1)^{\chi'} i \sqrt{\omega^4 - k^2 g^2} \cos a_{k'j'} \big)^{-1} \exp \Big(-\frac{\omega^2}{g} (r \sin a_{kj} + \varkappa \sin a_{k'j'}) - i \frac{M}{g} \sqrt{\omega^4 - k^2 g^2} \Big) \big) \big) \Big);$$

$$\eta^c(\varkappa) = \frac{f^0}{2\rho g} I_2^c \quad \text{for } u^* t < |M| < u_0 t, \tag{11}$$

$$I_2^c \simeq -2 \sum_{\alpha,\alpha'=0}^{1} D^{\alpha'} \Big(D \Big(I \sqrt{\frac{\pi g}{|M|}} \frac{\sqrt{q_\alpha}(k^2 + q_\alpha^2)^{3/4}}{|2k^2 - q_\alpha^2|^{1/2}} \Big(\sqrt{g \sqrt{k^2 + q_\alpha^2}} - (-1)^{\alpha'} \omega \Big)^{-1} \Big(\sqrt{k^2 + q_\alpha^2} \sin a_{k'j'} + (-1)^{\chi'} i q_\alpha \cos a_{k'j'} \Big)^{-1} \times$$
$$\times \exp \Big\{ -\sqrt{k^2 + q_\alpha^2} (r \sin a_{kj} + \varkappa \sin a_{k'j'}) + i \Big(\sqrt{g}\, t (k^2 + q_\alpha^2)^{1/4} - q_\alpha M + kz + (-1)^{\alpha+\alpha'} \frac{\pi}{4} \Big) \Big\} \Big) \Big), \quad q_\alpha \text{ — roots of the Eqn. } q \frac{\sqrt{g}\, t}{M} = 2(k^2 + q^2)^{3/4},$$

$$\eta^c(\varkappa) \to 0 \quad \text{for} \quad |M| > u_0 t,$$

where $M = (-1)^{\chi} r \cos a_{kj} + (-1)^{\chi'} \varkappa \cos a_{k'j'}$, $u^* = \frac{g}{2\omega^3} \sqrt{\omega^4 - k^2 g^2}$, $u_0 = \sqrt[4]{\frac{g^2}{108 k^2}}$ $(u^* < u_0)$; on moving lines parallel to the shoreline $(|M| = u_0 t)$,

$$\eta^c(\varkappa) = \frac{f^0}{2\rho g} I_3^c, \tag{12}$$

$$I_3^c \simeq -\frac{1}{3} \Gamma\Big(\frac{1}{3}\Big) (kR_\varkappa)^{-1/3} \sum_{\alpha=0}^{1} \frac{\sqrt{gk\sqrt{3}}}{\sqrt{gk\sqrt{3}} - (-1)^\alpha \omega} D^\alpha \Big(D \Big(I \big(\sqrt{3} \sin a_{k'j'} + (-1)^{\chi'} i \sqrt{2} \cos a_{k'j'} \big)^{-1} \exp \big(-\sqrt{3}\, k (r \sin a_{kj} + \varkappa \sin a_{k'j'}) \big) \exp \Big\{ i \Big(2\sqrt{2}\, kM + kz + (-1)^\alpha \frac{\pi}{6} \Big) \Big\} \Big) \Big)$$

$$(R_\varkappa = \sqrt{(r - \varkappa)^2 + z^2}, \ \Gamma \text{ — gamma function});$$

2) If $k_1 < k < \omega^2/g$, then the picture of the wave motion becomes more simple:

$$\eta^c(\varkappa) = \frac{f^0}{2\rho g} I_1^c \quad \text{for} \quad |M| < u^* t, \quad \eta^c(\varkappa) \to 0 \quad \text{for} \quad |M| > u^* t;$$

3) If $\omega^2 \leq gk$, which corresponds to the absence of poles in the integrand function in Eq. (7), then

$$\eta^c(\varkappa) = \frac{f^0}{2\rho g} I_2^c \quad \text{for} \quad |M| < u_0 t, \quad \eta^c(\varkappa) = \frac{f^0}{2\rho g} I_3^c \quad \text{for} \quad |M| = u_0 t, \quad \eta^c(\varkappa) \to 0 \quad \text{for} \quad |M| > u_0 t$$

($\eta^c(\kappa)$ represents the totality of m + 1 families, each of which consists of m + 1 straight lines and m + 1 reflections from the shore of quasiplanar waves propagating in the direction of increasing r). In contrast to the case of an infinitely deep fluid in which there is one straight line and one reflection of a wave from the vertical wall, a complex wave picture develops in the presence of an inclined bottom: the straight line and the reflected wave are converted into a family of waves, which move relative to one another and superpose onto one another. Furthermore, the wave picture of the motion depends on the relationship of parameters k and ω defining the system of pressures.

When $k < k_1$, there propagate in region $r > b$ nondecaying progressive waves (described by formulas (9), (10)) up to the moving boundary $r = b + u^*t$, behind which there propagate in the strip $u^*t < r - b < u_0 t$ ($u_0 > u^*$) waves (formulas (9), (11)), decaying according to $r^{-1/2}$, in whose background we can single-out straight lines $|M| = u_0 t$, on which the wave amplitude order $\sim r^{-1/3}$ (formulas (9), (12)). We refer to waves possessing a rectilinear front of this kind as divergent waves.

We now consider the case $k_1 < k < \omega^2/g$. Here, up to the moving boundary $r = b + u^*t$, there propagate progressive non-decaying waves; beyond the moving boundary there is an absence of wave motion. We shall assume that $k \geq \omega^2/g$. Then up to the boundary moving at the rate u_0 ($r = b + u_0 t$) there propagate progressive waves, decaying as $r^{-1/2}$; in this background we have movement of a divergent wave ($|M| = u_0 t$) on which the order of amplitude $\sim r^{-1/3}$. The wave picture closes with the divergent wave $r = b + u_0 t$, beyond which there is an absence of wave motion.

We consider waves of the discrete spectrum, localized in the coastal zone, which are obtained in explicit form determined from formula (8). When $k = \omega^2/(g \cos \ell\beta)$, singularities in formula (8), $1/(\sqrt{gk \cos \ell\beta} - \omega)$ and $1/(gk \cos \ell\beta - \omega^2)$, are mutually annihilating. For the coastal zone $D_\ell(\exp A_\ell r) \simeq D_\ell(1)$. In the zone of the shore, the Stokes wave is the defining wave for all waves of the discrete spectrum; the wave motion corresponding to this wave is described by the expression

$$\eta^d_{n-1} \simeq \frac{f^0}{\Delta\rho g}\frac{\varepsilon_{n-1}B^2_{1,n-1}}{\cos\beta}(\exp(-kb\cos\beta) - \exp(-ka\cos\beta)) \times$$
$$\times\left[\frac{\sqrt{gk\sin\beta}}{\sqrt{gk\sin\beta}-\omega}\exp i\left(kz - \sqrt{gk\sin\beta}\,t\right) + \frac{\sqrt{gk\sin\beta}}{\sqrt{gk\sin\beta}+\omega}\times\right.$$
$$\left.\times\exp i\left(kz + \sqrt{gk\sin\beta}\,t\right) - \frac{2\omega^2}{gk\sin\beta - \omega^2}\exp i(kz - \omega t)\right].$$

Waves of the discrete spectrum consist of three groups, one of which repeats the form of the traveling pressure, while the two remaining waves are stimulated by characteristic functions of the discrete spectrum and are directed oppositely to one another. In the coastal zone each of these groups consists of m terms, while in the zone of the shore the defining wave in the group is the Stokes wave.

As $t \to \infty$, a stationary regime of wave motion is formed. If $k < \omega^2/g$, nondecaying progressive waves propagate in the direction of increasing r when $r > b$; when $k \geq \omega^2/g$, the wave picture changes: in the background of waves decaying as $r^{-1/2}$ divergent waves propagate (order of amplitude $r^{-1/3}$).

APPENDIX

1. Notation used in formulas (2) and (3):

$$B_k^{(\chi)} = \frac{(-1)^{n-k}}{(p+s_c)^{n-1}}\prod_{\sigma=1}^{n-1}(p^2\sin\sigma\beta + q^2)^{-1/2}\prod_{\sigma=n-k+1}^{n-1}(p^2\sin\sigma^2\beta + q^2)\times$$
$$\times\prod_{\sigma=1}^{n-k}\operatorname{ctg}\sigma\beta\left[\frac{1}{2}p^2\sin 2\sigma\beta + (-1)^\chi iqs_c\right],$$
$$B_{kl} = (-1)^{n-k}\prod_{\sigma=n-k+1}^{n-1}\sin(\sigma - l)\beta\sin(\sigma+l)\beta\times$$
$$\times\prod_{\sigma=1}^{n-k}\operatorname{ctg}\sigma\beta\sin(\sigma+l)\beta\cos(\sigma-l)\beta,$$
$$\varepsilon_l = (-1)^{\frac{n+1}{2}}\left\{2(1+\cos l\beta)^{2(n-1)}\sin l\beta\prod_{\substack{\sigma=1\\ \sigma\neq l}}^{n-1}\sin(\sigma+l)\beta\sin(\sigma-l)\beta\right\}^{-1},$$
$$A_0 = -(s_c\sin a_{kj} + (-1)^\chi iq\cos a_{kj}),\quad A_l = -p\sin(a_{kj} + l\beta),$$
$$a_{kj} = 2(k-j)\beta,\quad s_c = \sqrt{p^2+q^2},\quad p = s_c\cos\lambda,\quad q = s_c\sin\lambda,$$
$$I = \exp\left\{i\frac{\pi}{4}(n-1)[(-1)^\chi + (-1)^{\chi'}]\right\}.$$

2. For operators D^0 and D^1 we agree that: for D^0 the summation extends over all indices for which $-(-1)^\chi \cos a_{kj} > 0$, for D^1 we assume $(-1)^\chi \cos a_{kj} > 0$.

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BOUNDARY OF MONOTONIC AND OSCILLATORY CONVECTIVE STABILITY OF A HORIZONTAL FLUID LAYER

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The problem of small oscillations of a heat-conducting fluid which occupies a horizontal layer and is close to mechanical equilibrium is examined here. It is assumed that the layer is heated from above, so that the fluid is stably stratified. As is known [1, 2], for sufficiently high viscosity, all modes are monotonically damped (the decrements are positive), but if the viscosity is low enough, then there are also oscillatory modes, which correspond to complex decrements with positive real and nonzero imaginary parts.

Here the limiting case of infinitely large Prandtl σ and Rayleigh R numbers is studied, the Grashoff number $G = R/\sigma$ being finite and fixed in value. The problem reduces to analysis of the spectral boundary-value problem for a fourth-order ordinary differential equation which is nonlinear in the spectral parameter, the decrement λ. The problem contains as auxiliary parameters the wavenumber α and G. For fixed α and G, it is easily established that there exists a countable set $\{\lambda_n\}_{n=1}^{\infty}$ of eigenvalues. In this case, the eigenvalues are all real if G is sufficiently small. When G, as it grows, reaches a definite critical value, there appear a series of pairs of complex-conjugate eigennumbers λ which, as usual, are determined from the appropriate transcendental equation. To analyze the equation, the method of one-dimensional perturbations (perturbations of boundary conditions) is applied. This method was used by Jeffries [3] in the convection problem. The method leads directly to an expansion of the left side of the transcendental equation in partial fractions, which facilitates study: specificaly, it helps in isolating the roots.

The minimum values in α of the critical Grashoff numbers G_n for the appropriate values of α and λ are determined. These are found separately for the even and odd modes with respect to the transverse variable. The asymptotes to G_n for $n \to \infty$ are constructed. It is remarkable that even for n = 1, the asymptotics yield good accuracy.

There are grounds for believing that the critical value of the Grashoff number $G_* = 729$, which results in the first appearance of an oscillatory mode, corresponds to the transition of turbulent convection at infinitely large Prandtl numbers [4].

1. Problem Statement. The stability spectrum ("spectrum of small oscillations") is determined in this case by the boundary-value problem

$$(D^2-\alpha^2)^2\varphi+\alpha^2R\theta=-\lambda(D^2-\alpha^2)\varphi; \quad (1.1)$$

$$(D^2-\alpha^2)\theta+\varphi=-\lambda\sigma\theta; \quad (1.2)$$

$$\varphi=\varphi'=\theta=0\ (z=\mp 1). \quad (1.3)$$

Here R is the Rayleigh number with a minus sign, so that positive R corresponds to stability; α^2 is the square of the modulus of the horizontal wave vector; D = d/dz; λ is the com-

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